ON THE MOTION OF GYROSCOPIC SYSTEMS

(DVIZHENIE GIROSKOPICHESKIKH SISTEM)

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V.S. NOVOSELOV
(Leningrad)

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In examining the motion of a gyroscopic system, we usually omit the second derivatives of Lagrangian coordinates in the linearized equations [1-6]. For the steady motion of gyroscopic systems in which damping is not present, such an operation was justified by Merkin [4]. In this paper we consider the case of a gyroscopic system with variable coefficients. The method suggested is based on ordinary substitution for the independent variables; this method is also of interest for the steady gyroscopic system.

1. The small free vibrations of stabilized gyroscopic systems on a fixed foundation, for small disturbances of the position coordinates q_1, ..., q_s and their derivatives, are given by equations

$$a_{ij}\ddot{q}_j + (b_{ij} + Hg_{ij})\dot{q}_j + c_{ij}q_j = 0 \tag{1}$$

in which a sufficiently large positive constant H denotes the angular momentum of the system with respect to the axes of the gyroscopes and $\| a_{ij} \|$ is the positive definite matrix of quadratic form. The coefficients in equation (1) are bounded time-functions and are of zero order with respect to H. The solutions of the stabilized system cannot have an order with respect to H greater than zero. The indices i and j take on the values of 1 to s; presence of two equal indices in a term means summation.

Let us consider the solution of the two systems of equations:

$$(b_{ij} + Hg_{ij})q_{j1} + c_{ij}q_{j1} = 0 \tag{2}$$

$$a_{ij}\ddot{q}_{j2} + (b_{ij} + Hg_{ij})\dot{q}_{j2} = 0 \tag{3}$$

If the determinant of the matrix of the gyroscopic terms $\| g_{ij} \|$ differs from zero, then for a sufficiently large value of H, the system (2) is solvable with respect to q_{j1}. The system (3) always has a solution with respect to q_{j2}.

We will show that the solution of system (1), satisfying the general conditions for $t = 0$, $q_i = q_i^0$, $\dot{q}_i = \dot{q}_i^0$ for any finite time interval $[0, T]$, is described in the coordinates q_{i1} and in the velocities $\dot{q}_{i2}$ with an accuracy to the order of H^{-1}. Then the system is subjected to the following initial conditions:

$$q_{i1}{}^\circ = q_i{}^\circ, \qquad [b_{ij}(0) + Hg_{ij}(0)]\dot{q}_{j1}{}^\circ + c_{ij}(0)\, q_j{}^\circ = 0 \tag{4}$$

$$q_{i2}{}^\circ = 0, \qquad \dot{q}_{i2}{}^\circ = \dot{q}_i{}^\circ - \dot{q}_{i1}{}^\circ \tag{5}$$

Introducing in equation (2) the new independent variables t/H and in equation (3) the variables Ht, denoting by (1) prime the derivatives with respect to the new independent variables, we obtain

$$\begin{gathered}(H^{-1}b_{ij} + g_{ij})\, q_{j1}' + c_{ij}\, q_{j1} = 0 \\ (H^{-1}b_{ij} + g_{ij})\, q_{j1}'' + (\dot{b}_{ij} + H\dot{g}_{ij} + c_{ij})\, q_{j1}' + H\dot{c}_{ij} q_{j1} = 0\end{gathered} \tag{6}$$

$$a_{ij}\, q_{j2}'' + (H^{-1}b_{ij} + g_{ij}) q_{j2}' = 0, \qquad q_{i2} = (f_{ij}\, q_{j2}')_0^t - \int_0^t \dot{f}_{ij} q_{j2}' dt \tag{7}$$

where f_{ij} is a bounded function of zero order with respect to the quantity H, and is obtained by the solution of the first group of equations (7) with respect to q_{i2}'. Derivatives f_{ij} cannot have an order higher than zero with respect to H.

Conventionally, we call the precession and nutation of the gyroscopic system (1) the motion described by $q_i^{(1)}$ and $q_i^{(2)}$, respectively, having initial conditions corresponding to equations (4) and (5). The solution of the system (1) with general initial conditions is equal to $q_i = q_i^{(1)} + q_i^{(2)}$ let $q_i^{(1)} = q_{i1} + x_i$, $q_i^{(2)} = q_{i2} + y_i$. The variables x_i and y_i are determined as a particular solution with zero initial conditions of the equations

$$a_{ij}\, \ddot{x}_j + (b_{ij} + Hg_{ij})\dot{x}_j + c_{ij} x_j = -H^{-2} a_{ij} q_{j1}'' \tag{8}$$

$$a_{ij}\, \ddot{y}_j + (b_{ij} + Hg_{ij})\dot{y}_j + c_{ij} y_j = -c_{ij} q_{j2} \tag{9}$$

As we can see from systems (6) and (7), for a finite time interval $[0, T]$ the variables q_{i1}'' have an order H and q_{j2} are of order H^{-1}. On the basis of equations (8) and (9) we obtain

$$\{x_i,\ \dot{x}_i,\ y_i,\ \dot{y}_i\} = O(H^{-1}) \tag{10}$$

The variables q_{i1} and q_{i2} have an order H^{-1}; then, using equation (10) we prove our assertion, namely

$$q_i = q_{i1} + O(H^{-1}), \qquad \dot{q}_i = \dot{q}_{i2} + O(H^{-1}) \tag{11}$$

Moreover, the quantity q_{i1} changes slowly, while q_{i2} changes as fast as $q_{i1} = 0(H^{-1})$; then equations (3) and (5) show that $\dot{q}_{i2} = 0(H)$.

2. If the system (1) is stable for a continuously applied disturbance and the variables q_{i1} and q_{i2} are bounded, then, as follows from the systems (6) (8) and (9), the result of equations (10) and (11) is valid for an infinite time interval.

Forced vibrations of the stabilized gyroscopic system, caused by the action of a generalized force $f_i(t)$, can be described with the particular solution, subjected to zero initial conditions, of the following system of equations:

$$(b_{ij} + Hg_{ij})\dot{q}_{j3} + c_{ij}q_{j3} = f_i(t)$$

The magnitude of the permitted error in the coordinates and in the velocities is determined by the solution of equation (8), subjected to zero initial conditions. The q_{j1}'' is replaced by the values obtained from solving the second group of equations (6) equated to Hf_i and subjected to corresponding initial conditions. Then, the error has to be of the order q_{j1}'', divided by H^2. If $f_i(t)$ has a zero order with respect to H, then the order of error is less than H^{-1} for any time interval in which $f_j(t)$ is bounded.

We will now give a certain refinement in the value of equations (10) and (11). Letting $\tau = Ht$ in equation (9), we have

$$a_{ij}\,y_j'' + (H^{-1}b_{ij} + g_{ij})\,y_j' = -\,H^{-2}c_{ij}\,(q_{j2} + y_j) \tag{12}$$

The quantity $q_{j2} + y_j$ has the order H^{-1}. To compute the particular solution of equation (12), with zero initial conditions we integrate with respect to τ from 0 to Ht; then the obtained particular solution, y_i', has the order of H^{-2}. Equation (12) leads to

$$y_i \doteq (f_{ij}\,y_j')_0^t - \int_0^t \dot{f}_{ij}y_j'dt + O(H^{-2}), \qquad y_i = O(H^{-2})$$

We say that the gyroscopic system satisfies condition (A) if g_{ij} and c_{ij} are equal to zero or have an order no greater than H^{-1}. As follows from system (6) q_{i1}'' in this case has a zero order with respect to H; then, on the basis of equation (8), we obtain $x_i = 0(H^{-2})$.

Hence, we have shown that the solution of equation (1) with arbitrary initial values satisfying conditions (A) is given by

3. Consider an automatically regulated gyroscopic system with supplementary k equations:

$$
\begin{aligned}
a_{ij}\ddot{q}_j + (b_{ij} + Hg_{ij})\dot{q}_j + c_{ij}q_j + d_{i\nu}\dot{q}_\nu + e_{i\nu}q_\nu &= 0 \\
A_{\mu\nu}\ddot{q}_\nu + B_{\mu\nu}\dot{q}_\nu + C_{\mu\nu}q_\nu + D_{\mu j}q_j &= 0
\end{aligned}
\tag{13}
$$

Here the indices i and j take the values of 1 to s; μ and ν of $s + 1$ to $s + k$; $\| a_{ij} \|$ and $\| A_{\mu\nu} \|$ are positive definite matrices of quadratic forms. Let the determinant of $\| g_{ij} \|$ not equal zero.

Denote by $\{ q_{i1}, q_{\mu 1} \}$ the solution obtained from equation (13), with the initial conditions

$$
\begin{gathered}
q_{i1}{}^\circ = q_i{}^\circ,\quad q_{\mu 1}{}^\circ = q_\mu{}^\circ, \quad \dot{q}_{\mu 1}{}^\circ = \dot{q}_\mu{}^\circ \\
[b_{ij}(0) + Hg_{ij}(0)]\,\dot{q}_{j1}{}^\circ + c_{ij}(0)\,q_{j1}{}^\circ + d_{i\nu}(0)\,\dot{q}_{\nu 1}{}^\circ + e_{i\nu}(0)\,q_{\nu 1}{}^\circ = 0
\end{gathered}
\tag{14}
$$

for $a_{ij} = 0$. Conventionally, we call $\{ q_i{}^{(1)}, q_\mu{}^{(1)} \}$ precession, which represents motion obtained by the solution of the system equation (13), with initial conditions, equation (14). We call the corresponding mutation the solution of the system, equation (13), subjected to the initial conditions:

$$
q_i{}^{\circ(2)} = q_\mu{}^{\circ(2)} = \dot{q}_\mu{}^{\circ(2)} = 0, \qquad \dot{q}_i{}^{\circ(2)} = \dot{q}_i^\circ - \dot{q}_{i1}{}^\circ
\tag{15}
$$

Therefore, we have

$$
q_i = q_i{}^{(1)} + q_i{}^{(2)}, \qquad q_\mu = q_\mu{}^{(1)} + q_\mu{}^{(2)}
$$

Let

$$
\begin{gathered}
q_i{}^{(1)} = q_{i1} + u_i, \quad q_\mu{}^{(1)} = q_{\mu 1} + u_\mu \\
q_i{}^{(2)} = q_{i2} + vi, \quad q_\mu{}^{(2)} = v_\mu
\end{gathered}
$$

Where q_{i2} is the same as in Section 1. We find the unknowns u_i, u_μ and v_i, v_μ and also their derivatives with respect to time, as the particular solutions with zero initial values of a corresponding system, which differs from the system, equation (13), by the presence of the right-hand side in the first s equations of a non-homogeneous term. In the first case we have the term $a_{ij}q_{j1}$ and in the second $c_{ij}v_j$. Repeating the conclusion of Section 1, we obtain

$$
\{ \dot{q}_{i1}\,;\ u_i,\ u_\mu,\quad \dot{u}_i,\quad \dot{u}_\mu;\quad v_i,\ v_\mu,\quad \dot{v}_i,\quad \dot{v}_\mu \} = O(H^{-1})
$$

Hence, we have

$$
\begin{aligned}
q_i &= q_{i1} + O(H^{-1}), \qquad & q_\mu &= q_{\mu 1} + O(H^{-1}) \\
q_i &= \dot{q}_{i1} + O(H^{-1}), \qquad & q_\mu &= q_{\mu 1} + O(H^{-1})
\end{aligned}
$$

Forced vibration of the stable system differing from the system of equation (13), by the presence on the right side of corresponding generalized forces $f_i(t)$ and $f_\mu(t)$, can be described by the particular solution of the following equations

$$(b_{ij} + Hg_{ij})\,\dot{q}_{j3} + c_{ij}\,q_{j3} + d_{i\nu}\,\dot{q}_{\nu 3} + e_{i\nu}\,q_{\nu 3} = f_i(t)$$

$$A_{\mu\nu}\,\ddot{q}_{\nu 3} + B_{\mu\nu}\,\dot{q}_{\nu 3} + C_{\mu\nu}q_{\nu 3} + D_{\mu j}\,q_{j3} = f_\mu(t)$$

The permitted error, as follows from the discussion in Section 2, has the order of H^{-1}.

BIBLIOGRAPHY

1. Krylov, A.N., K teorii giroskopa Anshiuttsa, izlozhennoi prof. Gekkelerom (On the theory of the Anschutz gyroscope put forward by Professor Gekkeler). *Izv. AN SSSR, ser. geog.: geofiz.* No. 4, 1940.

2. Roitenberg, Ya.N., Mnogogiroskopnaia vertikal' (The multigyroscopic vertical). *PMM* Vol. 10, No. 1, 1946.

3. Koshliakov, V.N., O deviatsiiakh girovertikali pri peremennoi skorosti sobstvennogo vrashe heniia giroskopa (On the deviation of the gyro-vertical by variation in velocity of rotation of the fundamental gyroscope). *Inzh. sbornik* Vol. 6, 1950.

4. Merkin, D.R., *Giroskopicheskie sistemy (Gyroscopic systems)*. GITTL, M., 1956.

5. Ishlinskii, A.Yu., K teorii girogorizontkompasa (On the theory of the gyrohorizontal compass). *PMM* Vol. 20, No. 4, 1956.

6. Ishlinskii, A.Yu., Teoriia dvukhgiroskopnoi vertikali (The theory of a two-gyroscope vertical). *PMM* Vol. 21, No. 2, 1957.

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